Appendix I

NONRADIATION CONDITION

DERIVATION OF THE CONDITION FOR NONRADIATION

The condition for radiation by a moving point charge given by Haus [1] is that its spacetime Fourier transform does possess components that are synchronous with waves traveling at the speed of light. Conversely, it is proposed that the condition for nonradiation by an ensemble of moving charge that comprises a charge-density function is that its spacetime Fourier transform does NOT possess components that are synchronous with waves traveling at the speed of light. The Haus derivation applies to a moving charge-density function as well because charge obeys superposition. The Haus derivation is summarized below.

The Fourier components of the current produced by the moving charge are derived. The electric field is found from the vector equation in Fourier space ($\mathbf{k}$, ω-space). The inverse Fourier transform is carried over the magnitude of $\mathbf{k}$. The resulting expression demonstrates that the radiation field is proportional to $\mathbf{J}_{\perp}(\frac{\omega}{c}\mathbf{n},\omega)$ where $\mathbf{J}_{\perp}(\mathbf{k},\omega)$ is the spacetime Fourier transform of the current perpendicular to $\mathbf{k}$ and $\mathbf{n} \equiv \frac{\mathbf{k}}{|k|}$. Specifically,

$$\mathbf{E}_{\perp}(\mathbf{r},\omega)\frac{d\omega}{2\pi} = \frac{c}{2\pi}\int \rho(\omega,\Omega)d\omega d\Omega \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{n}\ X\ \left(\mathbf{n}\ X\ \mathbf{J}_{\perp}\left(\frac{\omega}{c}\mathbf{n},\omega\right)e^{i\left(\frac{\omega}{c}\right)\mathbf{n}\bullet\mathbf{r}}\right) \tag{1}$$

The field $\mathbf{E}_{\perp}(\mathbf{r},\omega)\frac{d\omega}{2\pi}$ is proportional to $\mathbf{J}_{\perp}\left(\frac{\omega}{c}\mathbf{n},\omega\right)$, namely, the Fourier component for which $\mathbf{k} = \frac{\omega}{c}$. Factors of ω that multiply the Fourier component of the current are due to the density of modes per unit volume and unit solid angle. An unaccelerated charge does not radiate in free space, not because it experiences no acceleration, but because it has no Fourier component $\mathbf{J}_{\perp}\left(\frac{\omega}{c}\mathbf{n},\omega\right)$.

SPACETIME FOURIER TRANSFORM OF THE ELECTRON FUNCTION

The electron charge-density (mass-density) function is the product of a radial delta function ($f(r) = \frac{1}{r^2}\delta(r - r_n)$), two angular functions (spherical harmonic functions), and a time-harmonic function. The spacetime Fourier transform of the spherical current membrane in three dimensions in spherical coordinates plus time is given [2, 3] as follows:

$$M(s,\Theta,\Phi,\omega) = \int_0^{\infty}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \rho(r,\theta,\phi,t)\exp(-i2\pi sr[\cos\Theta\cos\theta + \sin\Theta\sin\theta\cos(\phi-\Phi)])\exp(-i\omega t)r^2\sin\theta d\phi d\theta drdt \tag{2}$$

With circular symmetry [2]

$$M(s,\Theta,\omega) = 2\pi\int_0^{\infty}\int_0^{\infty}\int_0^{\pi} \rho(r,\theta,t)J_0\left(2\pi sr\sin\Theta\sin\theta\right)\exp\left(-i2\pi sr\cos\Theta\cos\theta\right)r^2\sin\theta\exp\left(-i\omega t\right)d\theta drdt \tag{3}$$

With spherical symmetry [2],

$$M(s,\omega) = 4\pi\int_0^{\infty}\int_0^{\infty} \rho(r,t)\mathrm{sinc}(2sr)r^2\exp(-i\omega t)drdt \tag{4}$$

The functions that model the electron charge density are separable.

$$\rho(r,\theta,\phi,t) = f(r)g(\theta)h(\phi)k(t) \tag{5}$$

The orbitsphere function is separable into a product of functions of independent variables, $r, \theta, \phi,$ and t. The radial function, that satisfies the boundary condition is a delta function. The time functions are of the form $e^{i\omega t}$, the angular functions are spherical harmonics, sine or cosine trigonometric functions or sums of these functions, each raised to various powers. The spacetime Fourier transform is derived of the separable variables for the angular space function of $\sin\phi$ and $\sin\theta$. It follows from the spacetime Fourier transform given below that other possible spherical harmonic angular functions give the same form of result as the transform of $\sin\theta$ and $\sin\phi$. Using Eq. (4), $F(s)$, the space Fourier transform of $f(r) = \frac{1}{r^2}\delta(r - r_n)$ is given as follows:

$$F(s) = 4\pi\int_0^\infty \frac{1}{r^2}\delta(r-r_n)\operatorname{sinc}(2sr)r^2 dr \tag{6}$$

$$F(s) = 4\pi\operatorname{sinc}(2sr_n) \tag{7}$$

The subscript n is used hereafter; however, the quantization condition appears in the Excited States of the One-Electron Atom (Quantization) section. Quantization arises as "allowed" Maxwellian solutions corresponding to a resonance between the electron and a photon.

Using Eq. (3), $G_1^1(s,\Theta)$, the space Fourier transform of $g(\theta) = \sin\theta$ is given as follows where there is no dependence on ϕ:

$$G_1^1(s,\Theta) = 2\pi\int_0^\infty\int_0^\pi \sin\theta J_0\left(2\pi sr\sin\Theta\sin\theta\right)\exp\left(-i2\pi sr\cos\Theta\cos\theta\right)\sin\theta r^2 d\theta dr \tag{8}$$

$$G_1^1(s,\Theta) = 2\pi\int_0^\infty\int_0^\pi r^2\sin^2\theta J_0\left(2\pi sr\sin\Theta\sin\theta\right)\cos\left(2\pi sr\cos\Theta\cos\theta\right)d\theta dr \tag{9}$$

From Luke [4] and Abramowitz and Stegun [5]:

$$J_\upsilon(z) = \left(\frac{1}{2}z\right)^\nu\sum_{n=0}^\infty\frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{n!\Gamma(\nu+n+1)} = \left(\frac{1}{2}z\right)^\nu\sum_{n=0}^\infty\frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{n!(\nu+n)!} \tag{10}$$

Let

$$z = 2\pi sr\sin\Theta\sin\theta \tag{11}$$

With the substitution of Eqs. (11) and (10) into Eq. (9),

$$G_1^1(s,\Theta) = 2\pi\int_0^\infty\int_0^\pi r^2\sin^2\theta\left[\sum_{n=0}^\infty\frac{(-1)^n\left(\pi sr\sin\Theta\sin\theta\right)^{2n}}{n!n!}\right]\cos(2\pi sr\cos\Theta\cos\theta)d\theta dr \tag{12}$$

$$G_1^1(s,\Theta) = 2\pi\int_0^\infty r^2\int_0^\pi\sum_{n=0}^\infty\frac{(-1)^n\left(\pi sr\sin\Theta\right)^{2n}}{n!n!}\sin^{2(n+1)}\theta\cos\left(2\pi sr\cos\Theta\cos\theta\right)d\theta dr \tag{13}$$

$$G_1^1(s,\Theta) = 2\pi\int_0^\infty r^2\int_0^\pi\sum_{n=1}^\infty\frac{(-1)^{n-1}\left(\pi sr\sin\Theta\right)^{2(n-1)}}{(n-1)!(n-1)!}\sin^{2n}\theta\cos(2\pi sr\cos\Theta\cos\theta)d\theta dr \tag{14}$$

From Luke [6], with $\mathrm{Re}(\upsilon) > -\frac{1}{2}$:

$$J_\upsilon(z) = \frac{\left(\frac{1}{2}z\right)^\upsilon}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}\int_0^\pi\cos\left(z\cos\theta\right)\sin^{2\upsilon}\theta d\theta \tag{15}$$

Let

$$z = 2\pi sr\cos\theta \text{ and } n = \upsilon \tag{16}$$

Applying the relationship, the integral of a sum is equal to the sum of the integrals to Eq. (14), and transforming Eq. (14) into the form of Eq. (15) by multiplication by:

$$1 = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)\left(\pi sr\cos\Theta\right)^\upsilon}{\left(\pi sr\cos\Theta\right)^\upsilon\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)} \tag{17}$$

and by moving the constant outside of the integral gives:

$$G_1^1(s,\Theta)=2\pi\int_0^\infty r^2\sum_{\upsilon=1}^\infty\int_0^\pi \frac{(-1)^{\upsilon-1}\left(\pi sr\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\;\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)\left(\pi sr\cos\Theta\right)^{\upsilon}}{\left(\pi sr\cos\Theta\right)^{\upsilon}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}\sin^{2\upsilon}\theta\cos(2\pi sr\cos\Theta\cos\theta)d\theta dr \quad (18)$$

$$G_1^1\left(s,\Theta\right)=2\pi\int_0^\infty r^2\sum_{\upsilon=1}^\infty \frac{(-1)^{\upsilon-1}\left(\pi sr\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\;\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)\left(\pi sr\cos\Theta\right)^{\upsilon}}{\left(\pi sr\cos\Theta\right)^{\upsilon}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}\int_0^\pi\sin^{2\upsilon}\theta\cos(2\pi sr\cos\Theta\cos\theta)d\theta dr \quad (19)$$

Applying Eq. (15),

$$G_1^1\left(s,\Theta\right)=2\pi\int_0^\infty r^2\sum_{\upsilon=1}^\infty \frac{(-1)^{\upsilon-1}\left(\pi sr\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}{\left(\pi sr\cos\Theta\right)^{\upsilon}}J_\upsilon\left(2\pi sr\cos\Theta\right)dr \quad (20)$$

Collecting the r raised to a power terms, Eq. (20) becomes,

$$G_1^1(s,\Theta)=2\pi\sum_{\upsilon=1}^\infty\int_0^\infty \frac{(-1)^{\upsilon-1}\left(\pi s\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}{\left(\pi s\cos\Theta\right)^{\upsilon}}r^{\upsilon}J_\upsilon(2\pi sr\cos\Theta)dr \quad (21)$$

Let $r=\dfrac{r'}{2\pi\cos\Theta}$; $dr=\dfrac{dr'}{2\pi\cos\Theta}$,

$$G_1^1(s,\Theta)=2\pi\sum_{\upsilon=1}^\infty\int_0^\infty \frac{(-1)^{\upsilon-1}\left(\pi s\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}{\left(\pi s\cos\Theta\right)^{\upsilon}\left(2\pi\cos\Theta\right)^{\upsilon+1}}r'^{\upsilon}J_\upsilon(sr')dr' \quad (22)$$

Consider the Hankel transform formula from Bateman [7]:

$$\begin{aligned}&\int_0^\infty r^{\upsilon}J_\upsilon(rs)dr\\ &=s^{-\left(\frac{1}{2}\right)}\int_0^\infty r^{\upsilon-\frac{1}{2}}\left(rs\right)^{\left(\frac{1}{2}\right)}J_\upsilon(rs)dr\\ &=2^{\upsilon-1}\pi^{\left(\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}+\upsilon\right)s^{-\upsilon}\left[J_\upsilon\left(s\right)\mathbf{H}_{\upsilon-1}\left(s\right)-\mathbf{H}_\upsilon\left(s\right)J_{\upsilon-1}\left(s\right)\right]\end{aligned} \quad (23)$$

where the radius is normalized to the dimensionless parameter r that satisfies the conditions,

$$\begin{aligned}&r^{\upsilon-\frac{1}{2}},\ 0<r<1\\ &0,\ r>1\\ &\mathrm{Re}\,\upsilon>-\frac{1}{2}\end{aligned} \quad (24)$$

By applying Eq. (23), Eq. (22) becomes,

$$G_1^1(s,\Theta)=2\pi\sum_{\upsilon=1}^\infty\frac{(-1)^{\upsilon-1}\left(\pi s\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}{\left(\pi s\cos\Theta\right)^{\upsilon}\left(2\pi\cos\Theta\right)^{\upsilon+1}}2^{\upsilon-1}\pi^{\left(\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}+\upsilon\right)s^{-\upsilon}\begin{bmatrix}J_\upsilon\left(s\right)\mathbf{H}_{\upsilon-1}\left(s\right)\\ -\mathbf{H}_\upsilon\left(s\right)J_{\upsilon-1}\left(s\right)\end{bmatrix} \quad (25)$$

By collecting power terms of s gives

$$G_1^1(s,\Theta)=2\pi\sum_{\upsilon=1}^\infty\frac{(-1)^{\upsilon-1}\left(\pi\sin\Theta\right)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}{\left(\pi\cos\Theta\right)^{2\upsilon+1}2^{\upsilon+1}}2^{\upsilon-1}\pi^{\left(\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}+\upsilon\right)s^{-2}\left[J_\upsilon\left(s\right)\mathbf{H}_{\upsilon-1}\left(s\right)-\mathbf{H}_\upsilon\left(s\right)J_{\upsilon-1}\left(s\right)\right] \quad (26)$$

Next, $H_1^1(s,\Theta,\Phi)$, the space Fourier transform of $h(\phi)=\sin\phi$, is considered wherein the radius is normalized to the dimensionless parameter r as given in Eq. (24). Using Eq. (2) $H_1^1(s,\Theta,\Phi)$ is

$$H_1^1(s,\Theta,\Phi)=\int_0^\pi\int_0^{2\pi}\int_0^1\sin\phi\exp(-i2\pi sr[\cos\Theta\cos\theta+\sin\Theta\sin\theta\cos(\phi-\Phi)])r^2\sin\theta drd\phi d\theta \quad (27)$$

By setting

$$\alpha = \alpha(s,\theta,\phi,\Theta,\Phi) = 2\pi s[\cos\Theta\cos\theta + \sin\Theta\sin\theta\cos(\phi-\Phi)] \tag{28}$$

Eq. (28) simplifies to:

$$H_1^1(s,\Theta,\Phi) = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \sin\phi\sin\theta e^{-i\alpha r} r^2 dr d\phi d\theta \tag{29}$$

Following the radial integration [8], $H_1^1(s,\Theta,\Phi)$ is:

$$H_1^1(s,\Theta,\Phi) = \int_0^{\pi}\int_0^{2\pi} \sin\phi\sin\theta\left[\frac{2\cos\alpha}{\alpha^2} + \frac{\sin\alpha}{\alpha} - \frac{2\sin\alpha}{\alpha^3} + i\left(\frac{\cos\alpha}{\alpha} - \frac{2\cos\alpha}{\alpha^3} - \frac{2\sin\alpha}{\alpha^2} + \frac{2}{\alpha^3}\right)\right] d\phi d\theta \tag{30}$$

Based on the spatial similarity of $h(\phi) = \sin\phi$ and $g(\theta) = \sin\theta$, the respective Fourier transforms are similar and considered nonzero since the inverse Fourier transforms are the original trigonometric functions.

The time Fourier transform of $q(t) = \mathrm{Re}\{\exp(i\omega_n t)\}$ is given as follows [3]:

$$Q(\omega) = \int_0^{\infty} \cos\omega_n t \exp(-i\omega t) dt = \frac{1}{2\pi}\frac{1}{2}[\delta(\omega-\omega_n) + \delta(\omega+\omega_n)] \tag{31}$$

where ω_n is the angular frequency given by Eq. (1.36) corresponding to the frequency of a potentially emitted photon as given in Chp. 2.

A very important theorem of Fourier analysis states that the Fourier transform of a product is the convolution of the individual Fourier transforms [9]. By applying this theorem, the spacetime Fourier transform of an orbitsphere, $M_\ell^{m_\ell}(s,\Theta,\Phi,\omega)$ is of the following form:

$$M_\ell^{m_\ell}(s,\Theta,\Phi,\omega) = F(s) \otimes G_\ell^{m_\ell}(s,\Theta) \otimes H_\ell^{m_\ell}(s,\Theta,\Phi) \otimes Q(\omega) \tag{32}$$

Therefore, the spacetime Fourier transform, $M_1^1(s,\Theta,\Phi,\omega)$, is the convolution of Eqs. (7), (26), and (30-31).

$$\begin{aligned} M_1^1(s,\Theta,\Phi,\omega) &= 4\pi \mathrm{sinc}(2sr_n) \otimes H_1^1(s,\Theta,\Phi) \\ &\otimes 2\pi\sum_{\upsilon=1}^{\infty}\left\{ 2\pi\sum_{\upsilon=1}^{\infty} \begin{array}{l} \dfrac{(-1)^{\upsilon-1}(\pi\sin\Theta)^{2(\upsilon-1)}}{(\upsilon-1)!(\upsilon-1)!}\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{1}{2}\right)}{(\pi\cos\Theta)^{2\upsilon+1}2^{\upsilon+1}} \\ \cdot 2^{\upsilon-1}\pi^{\left(\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}+\upsilon\right)s^{-2}\left[J_\upsilon(s)\mathbf{H}_{\upsilon-1}(s) - \mathbf{H}_\upsilon(s)J_{\upsilon-1}(s)\right] \end{array} \right\} \\ &\otimes \frac{1}{4\pi}[\delta(\omega-\omega_n) + \delta(\omega+\omega_n)] \end{aligned} \tag{33}$$

The spherical harmonics functions are:

$$Y_\ell^m(\theta,\phi) = N_{\ell,m}P_\ell^m(\cos\theta)e^{im\phi} \tag{34}$$

Generalizing the exemplary functions $\sin\theta$ and $\sin\phi$, the Fourier transforms of the spherical harmonics expressed in terms of the respective integrals are given by:

$$G_\ell^{m_\ell}(s,\Theta) = 2\pi N_{\ell,m}\int_0^{\infty}\int_0^{\pi} P_\ell^m(\cos\theta) J_0(2\pi sr\sin\Theta\sin\theta)\exp(-i2\pi sr\cos\Theta\cos\theta)\sin\theta r^2 d\theta dr \tag{35}$$

and

$$H_\ell^{m_\ell}(s,\Theta,\Phi) = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{im\phi}\exp(-i2\pi sr[\cos\Theta\cos\theta + \sin\Theta\sin\theta\cos(\phi-\Phi)])r^2\sin\theta d\phi d\theta dr \tag{36}$$

In the general case, the spacetime Fourier transform, $M_\ell^{m_\ell}(s,\Theta,\Phi,\omega)$, is the convolution of Eqs. (7), (31), and (35-36).

$$M_\ell^{m_\ell}(s,\Theta,\Phi,\omega) = 4\pi\mathrm{sinc}(2sr_n) \otimes G_\ell^{m_\ell}(s,\Theta) \otimes H_\ell^{m_\ell}(s,\Theta,\Phi) \otimes \frac{1}{4\pi}[\delta(\omega-\omega_n) + \delta(\omega+\omega_n)] \tag{37}$$

wherein $G_\ell^{m_\ell}(s,\Theta)$ and $H_\ell^{m_\ell}(s,\Theta,\Phi)$ are the spherical-coordinate Fourier transforms of $N_{\ell,m}P_\ell^m(\cos\theta)$ and $e^{im\phi}$, respectively. The condition for nonradiation of a moving charge-density function is that the spacetime Fourier transform of the current-density function must not have waves synchronous with waves traveling at the speed of light, that is synchronous with $\frac{\omega_n}{c}$ or synchronous with $\frac{\omega_n}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}}$ where ε is the dielectric constant of the medium. The Fourier transform of the charge-density function of the orbitsphere (membrane bubble of radius r) is given by Eq. (37). In the case of time-harmonic motion, the current-density function is given by the time derivative of the charge-density function. Thus, the current-density function is given by the product of the constant angular velocity and the charge-density function. The Fourier transform of the current-density function of the orbitsphere is given by the product of the constant angular velocity and Eq. (37). Consider the radial and time parts of $K_\ell^{m_\ell}(s,\Theta,\Phi,\omega)$, the Fourier transform of the current-density function, where the angular transforms $G_\ell^{m_\ell}(s,\Theta)\otimes H_\ell^{m_\ell}(s,\Theta,\Phi)$ are taken as not zero:

$$K_\ell^{m_\ell}(s,\Theta,\Phi,\omega)=4\pi\omega_n\frac{\sin(2sr_n)}{2sr_n}\otimes G_\ell^{m_\ell}(s,\Theta)\otimes H_\ell^{m_\ell}(s,\Theta,\Phi)\otimes\frac{1}{4\pi}[\delta(\omega-\omega_n)+\delta(\omega+\omega_n)] \tag{38}$$

For the case that the current-density function is constant corresponding to $Y_0^0(\theta,\phi)$, the proceeding factor ω_n of the RHS of Eq. (38) is zero. For time harmonic motion, with angular velocity, ω, Eq. (38) is nonzero only for $\omega=\omega_n$; thus, $-\infty<s<\infty$ becomes finite only for the corresponding wavenumber, s_n. The relationship between the radius and the wavelength is:

$$v_n=\lambda_n f_n \tag{39}$$

$$v_n=2\pi r_n f_n=\lambda_n f_n \tag{40}$$

$$2\pi r_n=\lambda_n \tag{41}$$

Radiation of the bound electron requires an excited state wherein a potentially emitted photon circulates along the orbitsphere at light speed. The nature of an excited state as shown in the Excited States of the One-Electron Atom (Quantization) section is a superposition of an electron and a photon comprising two-dimensional shells of current and field lines, respectively, at the same radius as defined by $\delta(r-r_n)$[1]. Due to the further nature of the photon possessing light-speed angular motion, the electron motion and corresponding spatial and temporal parameters may be considered relative to light-speed for the laboratory frame of the electron's constant angular velocity. A radial correction exists due to Special Relativistic effects. Consider the wave vector of the sinc function. When the velocity is c corresponding to a potentially emitted photon,

$$\mathbf{s}_n\bullet\mathbf{v}_n=\mathbf{s}_n\bullet\mathbf{c}=\omega_n \tag{42}$$

the relativistically corrected wavelength given by Eq. (1.279) is[2]:

$$\lambda_n=r_n \tag{43}$$

The charge-density functions in spherical coordinates plus time are given by Eqs. (1.27-1.29). In the case of Eq. (1.27), the wavelength of Eq. (42) is independent of θ; whereas, in the case of Eqs. (1.28-1.29), the wavelength in Eq. (42) is a function of $\sin\theta$. Thus, in the latter case, Eq. (43) holds wherein the relationship of wavelength and the radius as a function of θ are given by $r_n\sin\theta=\lambda_n\sin\theta$.

Substitution of Eq. (43) into the sinc function (Eq. (38)) results in the vanishing of the entire Fourier transform of the current-density function. Thus, spacetime harmonics of $\frac{\omega_n}{c}=k$ or $\frac{\omega_n}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}}=k$ do not exist for which the Fourier transform of the current-density function is nonzero. Radiation due to charge motion does not occur in any medium when this boundary condition is met. Note that the boundary condition for the solution of the radial function of the hydrogen atom with the Schrödinger equation is $\Psi\to 0$ as $r\to\infty$. Here, however, the boundary condition is derived from Maxwell's equations: For non-radiative states, the current-density function must not possess spacetime Fourier components that are synchronous with waves traveling at the speed of light. An alternative derivation to that of Haus [1] considering the macro-Maxwellian case and boundary conditions that provides acceleration without radiation is given by Abbott [10].

NONRADIATION BASED ON THE ELECTROMAGNETIC FIELDS AND THE POYNTING POWER VECTOR

A point charge undergoing periodic motion accelerates and as a consequence radiates power P according to the Larmor formula:

$$P=\frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3}a^2 \tag{44}$$

where e is the charge, a is its acceleration, ε_0 is the permittivity of free space, and c is the speed of light. Although an accelerated *point* particle radiates, an *extended distribution* modeled as a superposition of accelerating charges does not have to radiate [1, 10-13]. An ensemble of charges, all oscillating at the same frequency, create a radiation pattern with a number of

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[1] Note that the equations of exited state photons given by Eq. (2.15) are not the macro-Maxwellian spherical resonator cavity solutions. The latter is the superposition of many photons comprising a three-dimensional electromagnetic wave in the cavity with the associated macro-boundary conditions. Haus [1] does not address the quantization of single-photon radiation of a bound state that conserves the angular momentum of the photon and single bound electron based on their respective natures. However, the superposition of many photons obeying the quantization condition on a single electron converges to the macro-Maxwellian result. Haus considers an example of rectilinear oscillation of a free point charge that would radiate many photons of many frequencies. It is the macro-Maxwellian case and boundary conditions that Haus addresses in his paper [1] on radiation from point charges. Since Maxwell's equations are obeyed on all scales, the converse of the condition for radiation gives rise to the condition of nonradiation of the bound electron.

[2] In the frame synchronous with waves traveling at the speed of light, the lab-frame electron motion is on a sphere with a radius contracted by the factor 2π. The derivation is given in the Special Relativistic Effect on the Electron Radius and the Relativistic Ionization Energies section. With the wavelength in the speed of light frame given by Eq. (43), the relativistic invariance of the angular momentum of the electron of $\hbar$ (Eq. (1.37)) provides that the corresponding relativistic electron mass (integral of the mass density over the surface) is $2\pi m_e$.

nodes. The same applies to current patterns in phased array antenna design [14]. It is possible to have an infinite number of charges oscillating in such as way as to cause destructive interference or nodes in all directions. The electromagnetic far field is determined from the current distribution in order to obtain the condition, if it exists, that the electron current distribution given by Eq. (49) must satisfy such that the electron does not radiate.

The charge-density functions of the electron orbitsphere in spherical coordinates plus time are given by Eqs. (1.27-1.29). For $\ell = 0$, $N = \frac{-e}{8\pi r_n^2}$, and the charge-density function is:

ℓ **= 0**

$$\rho(r,\theta,\phi,t) = \frac{e}{8\pi r_n^2}[\delta(r-r_n)]\left[Y_0^0(\theta,\phi) + Y_\ell^m(\theta,\phi)\right] \tag{45}$$

The equipotential, uniform or constant charge-density function (Eq. (1.27) and Eq. (49)) further comprises a current pattern given in the Orbitsphere Equation of Motion for $\ell = 0$ Based on the Current Vector Field (CVF) section. It also corresponds to the nonradiative $n = 1$, $\ell = 0$ state of atomic hydrogen and to the spin function of the electron. The current-density function is given by multiplying Eq. (47) by the modulation frequency corresponding to the constant angular velocity ω_n. There is acceleration without radiation, in this case, centripetal acceleration. A static charge distribution exists even though each point on the surface is accelerating along a great circle. Haus' condition predicts no radiation for the entire ensemble. The same result is trivially predicted from consideration of the fields and the radiated power. Since the current is not time dependent, the fields are given by:

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{46}$$

and

$$\nabla \times \mathbf{E} = 0 \tag{47}$$

which are the electrostatic and magnetostatic cases, respectively, with no radiation.

In cases of orbitals of heavier elements and excited states of one electron-atoms and atoms or ions of heavier elements that are not constant as given by Eqs. (1.28-1.29), the constant spin function is modulated by a time and spherical harmonic function. The modulation or traveling charge-density wave corresponds to an orbital angular momentum in addition to a spin angular momentum. These states are typically referred to as p, d, f, etc. orbitals and correspond to an ℓ quantum number not equal to zero. Haus' condition also predicts nonradiation for a constant spin function modulated by a time and spherically harmonic orbital function. However, in the case that such a state arises as an excited state by photon absorption, it is radiative due to a radial dipole term in its current-density function since it possesses spacetime Fourier transform components synchronous with waves traveling at the speed of light as given in the Instability of Excited States section.

The nonradiation condition given by Eqs. (38) and (42-43) may be confirmed by determining the fields and the current distribution condition that is nonradiative based on Maxwell's equations.

For $\ell \neq 0$, $N = \frac{-e}{4\pi r_n^2}$. The charge-density functions including the time-function factor are:

ℓ **≠ 0**

$$\rho(r,\theta,\phi,t) = \frac{e}{4\pi r^2}[\delta(r-r_n)]\left[Y_0^0(\theta,\phi) + \mathrm{Re}\left\{Y_\ell^m(\theta,\phi)e^{im\omega_n t}\right\}\right] \tag{48}$$

where $\mathrm{Re}\left\{Y_\ell^m(\theta,\phi)e^{im\omega_n t}\right\} = P_\ell^m(\cos\theta)\cos(m\phi + m\omega_n t)$. In the cases that $m \neq 0$, Eqs. (1.28-1.29) and Eq. (48) is a spherical harmonic traveling charge-density wave of quantum number m that moves on the surface of the orbitsphere about the z-axis at angular frequency ω_n and modulates the orbitsphere corresponding to $\ell = 0$ at $m\omega_n$. Since the charge is modulated time harmonically about the z-axis with the frequency $m\omega_n$ and the current-density function is given by the time derivative of the charge-density function, the current-density function is given by the normalized product of the constant modulation angular velocity and the charge-density function. The first current term of Eq. (48) is static. Thus, it is trivially nonradiative. The current due to the time dependent term is

$$\mathbf{J} = \frac{m\omega_n}{2\pi}\frac{e}{4\pi r_n^2} N[\delta(r-r_n)]\,\mathrm{Re}\left\{Y_\ell^m(\theta,\phi)\right\}\left[\mathbf{u}(t)\times\mathbf{r}\right]$$

$$= \frac{m\omega_n}{2\pi}\frac{e}{4\pi r_n^2} N[\delta(r-r_n)]\,\mathrm{Re}\left\{Y_\ell^m(\theta,\phi)e^{im\omega_n t}\right\}\left[\mathbf{u}\times\mathbf{r}\right]$$

$$= \frac{m\omega_n}{2\pi}\frac{e}{4\pi r_n^2} N'[\delta(r-r_n)]\,\mathrm{Re}\left(P_\ell^m(\cos\theta)e^{im\phi}e^{im\omega_n t}\right)\left[\mathbf{u}\times\mathbf{r}\right] \tag{49}$$

$$= \frac{m\omega_n}{2\pi}\frac{e}{4\pi r_n^2} N'[\delta(r-r_n)]\left(P_\ell^m(\cos\theta)\cos(m\phi+m\omega_n t)\right)\left[\mathbf{u}\times\mathbf{r}\right]$$

$$= \frac{m\omega_n}{2\pi}\frac{e}{4\pi r_n^2} N'[\delta(r-r_n)]\left(P_\ell^m(\cos\theta)\cos(m\phi+m\omega_n t)\right)\sin\theta\hat{\phi}$$

where N and N' are normalization constants. The vectors are defined as:

$$\hat{\phi} = \frac{\hat{u}\times\hat{r}}{|\hat{u}\times\hat{r}|} = \frac{\hat{u}\times\hat{r}}{\sin\theta}; \qquad \hat{u} = \hat{z} = orbital\ axis \tag{50}$$

$$\hat{\theta} = \hat{\phi}\times\hat{r} \tag{51}$$

"^" denotes the unit vectors $\hat{u} \equiv \frac{\mathbf{u}}{|\mathbf{u}|}$, non-unit vectors are designed in bold, and the current function is normalized. For time-varying electromagnetic fields, Jackson [15] gives a generalized expansion in vector spherical waves that are convenient for electromagnetic boundary-value problems possessing spherical symmetry properties and for analyzing multipole radiation from a localized source distribution. The Green function $G(\mathbf{x}',\mathbf{x})$ which is appropriate to the equation:

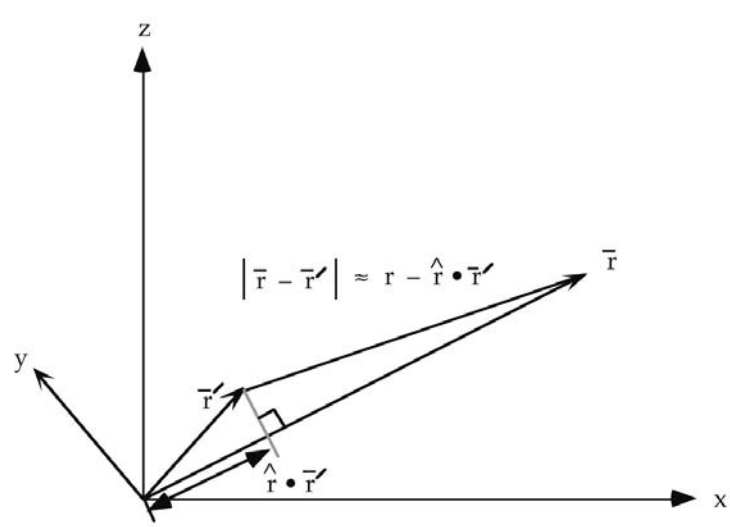
$$\left(\nabla^2 + k^2\right)G(\mathbf{x}',\mathbf{x}) = -\delta(\mathbf{x}'-\mathbf{x}) \tag{52}$$

in the infinite domain with the spherical wave expansion for the outgoing wave Green function is:

$$G(\mathbf{x}',\mathbf{x}) = \frac{e^{-ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik\sum_{\ell=0}^{\infty} j_\ell(kr_<)h_\ell^{(1)}(kr_>)\sum_{m=-\ell}^{\ell} Y_{\ell,m}^*(\theta',\phi')Y_{\ell,m}(\theta,\phi) \tag{53}$$

General spherical coordinates are shown in Figure AI.1.

Figure AI.1. Far field approximation.



Jackson [15] further gives the general multipole field solution to Maxwell's equations in a source-free region of empty space with the assumption of a time dependence $e^{i\omega t}$.

$$\mathbf{B} = \sum_{\ell,m}\left[a_E(\ell,m) f_\ell(kr)\mathbf{X}_{\ell,m} - \frac{i}{k} a_M(\ell,m)\nabla\times g_\ell(kr)\mathbf{X}_{\ell,m}\right]$$
$$\mathbf{E} = \sum_{\ell,m}\left[\frac{i}{k} a_E(\ell,m)\nabla\times f_\ell(kr)\mathbf{X}_{\ell,m} + a_M(\ell,m) g_\ell(kr)\mathbf{X}_{\ell,m}\right] \tag{54}$$

where the cgs units used by Jackson are retained in this section. The radial functions $f_\ell(kr)$ and $g_\ell(kr)$ are of the form:

$$g_\ell(kr) = A_\ell^{(1)} h_\ell^{(1)} + A_\ell^{(2)} h_\ell^{(2)} \tag{55}$$

$\mathbf{X}_{\ell,m}$ is the vector spherical harmonic defined by:

$$\mathbf{X}_{\ell,m}(\theta,\phi) = \frac{1}{\sqrt{\ell(\ell+1)}}\mathbf{L}Y_{\ell,m}(\theta,\phi) \tag{56}$$

where

$$\mathbf{L} = \frac{1}{i}(\mathbf{r}\times\nabla) \tag{57}$$

The coefficients $a_E(\ell,m)$ and $a_M(\ell,m)$ of Eq. (54) specify the amounts of electric (ℓ,m) multipole and magnetic (ℓ,m) multipole fields, and are determined by sources and boundary conditions as are the relative proportions in Eq. (55). Jackson gives the result of the electric and magnetic coefficients from the sources as:

$$a_E(\ell,m) = \frac{4\pi k^2}{i\sqrt{\ell(\ell+1)}}\int Y_\ell^{m*}(\theta,\phi)\left\{\rho\frac{\partial}{\partial r}\left[rj_\ell(kr)\right] + \frac{ik}{c}(\mathbf{r}\cdot\mathbf{J})\, j_\ell(kr) - ik\nabla\cdot(r\times\boldsymbol{M})\, j_\ell(kr)\right\}d^3x \tag{58}$$

and

$$a_M(\ell,m) = \frac{-4\pi k^2}{\sqrt{\ell(\ell+1)}}\int j_\ell(kr) Y_\ell^{m*}(\theta,\phi)\mathbf{L}\cdot\left(\frac{\mathbf{J}}{c} + \nabla\times\boldsymbol{M}\right)d^3x \tag{59}$$

respectively, where the distribution of charge $\rho(\mathbf{x},t)$, current $\mathbf{J}(\mathbf{x},t)$, and intrinsic magnetization $\boldsymbol{M}(\mathbf{x},t)$ are harmonically varying sources: $\rho(\mathbf{x})e^{-\omega t}$, $\mathbf{J}(\mathbf{x})e^{-\omega t}$, and $\boldsymbol{M}(\mathbf{x})e^{-\omega t}$. From Eq. (49), the charge and intrinsic magnetization terms are zero. Also, the current $\mathbf{J}(\mathbf{x},t)$ is in the $\hat{\phi}$ direction; thus, the $a_E(\ell,m)$ coefficient given by Eq. (58) is zero since $\mathbf{r}\cdot\mathbf{J} = 0$. Substitution of Eq. (49) into Eq. (59) gives the magnetic multipole coefficient $a_M(\ell,m)$:

$$a_M(\ell,m) = \frac{-4\pi k^2}{\sqrt{\ell(\ell+1)}}\int j_\ell(kr) Y_\ell^{m*}(\theta,\phi)\mathbf{L}\cdot\left(\frac{\frac{m\omega_n}{2\pi}\frac{e}{4\pi r_n^2}N\delta(r-r_n)Y_\ell^m(\theta,\phi)\sin\theta\hat{\phi}}{c}\right)d^3x \tag{60}$$

wherein the separable time harmonic function of the current is considered separately in Eq. (81). Each mass-density element of the electron moves about the z-axis along a circular orbit of radius $r_n\sin\theta$ in such a way that ϕ, changes at a constant rate. That is $\phi = \omega t$ at time t where $m\omega_n$ is the constant angular modulation frequency given in Eq. (49), and

$$r(t) = \mathbf{i}r_n\sin\theta\cos\omega t + \mathbf{j}r_n\sin\theta\sin\omega t \tag{61}$$

is the parametric equation of the circular orbit. The relationships between the Cartesian ($\mathbf{ijk}$) and spherical ($\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_\phi$) coordinates are [16]:

$$\mathbf{e}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$$
$$\mathbf{e}_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta \tag{62}$$
$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$$

The selection rules (Eq. (2.86)) for the conservation of angular momentum must be satisfied during the emission of a single photon of angular momentum $\hbar$:

$$\Delta\ell = \pm 1 \tag{63}$$

The photon's angular momentum given by Eq. (4.1) is:

$$\mathbf{m} = \int\frac{1}{8\pi c}\mathrm{Re}\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B}^*)\right]dx^4 = \hbar \tag{64}$$

requiring a matching change in the electron's angular momentum. With emission, the radius must decrease in order to conserve the photon's energy

$$E = \hbar\omega \tag{65}$$

and the electron's energy in the inverse-radius Coulomb potential:

$$V = \frac{-Ze^2}{4\pi\varepsilon_0 r} \tag{66}$$

The radial electric dipole current for a potentially emitted photon for the selection-rule condition of Eq. (2.86) given by Eq. (2.90) is

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{J} = J\mathbf{k} = J\left(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta\right) \tag{67}$$

Then, for radiation to occur from the rotating spherical harmonic current (Eq. (49)) while obeying the selection rules and the requirement of an allowed azimuthal-only $\mathbf{B}$ (Eq. (2.102)) pertaining to the emission of a single photon, the radiated magnetic field must have $\mathbf{e}_\phi$ only dependence. Further given Jackson's Eq. (16.84-16.89) [15] for the relationship of $a_M(\ell,m)$ to $\mathbf{B}$, the components of L in Eq. (60) are restricted to those in the xy-plane, the L_x and L_y components. It can easily be appreciated that this result also arises from application of $\mathbf{L}\cdot\mathbf{J}$ to Eq. (67) with the use of the vector identity given by Eq. (16.90) of Jackson [15]:

$$\mathbf{L}\cdot\mathbf{J} = i\nabla\cdot(\mathbf{r}\times\mathbf{J}) \tag{68}$$

Then, the nonradiation condition tests whether the components of the rotating spherical harmonic current that are parallel to those of Eq. (67) give rise to radiation.

Jackson gives the operator in the xy-plane corresponding to the current motion in this plane and the relations for $Y_\ell^m(\theta,\phi)$ [15]:

$$L_+ = L_x + iL_y = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \tag{69}$$

$$L_+Y_\ell^m(\theta,\phi) = \sqrt{(\ell-m)(\ell+m+1)}Y_\ell^{m+1}(\theta,\phi) \tag{70}$$

Using Eq. (69), $\mathbf{L}\cdot\mathbf{J}$ of Eq. (59) is

$$\begin{aligned} L_+\left(Y_\ell^m(\theta,\phi)\sin\theta\right) &= e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_\ell^m(\theta,\phi)\sin\theta \\ &= e^{i\phi}Y_\ell^m(\theta,\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\sin\theta + e^{i\phi}\sin\theta\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_\ell^m(\theta,\phi) \end{aligned} \tag{71}$$

Using Eq. (70) in Eq. (71) gives:

$$L_+\left(Y_\ell^m(\theta,\phi)\sin\theta\right) = e^{i\phi}Y_\ell^m(\theta,\phi)\cos\theta + \sin\theta\sqrt{(\ell-m)(\ell+m+1)}Y_\ell^{m+1}(\theta,\phi) \tag{72}$$

The spherical harmonic is given as

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi} = N_{\ell,m}P_\ell^m(\cos\theta)e^{im\phi} \tag{73}$$

Thus, Eq. (72) is given as:

$$L_+\left(Y_\ell^m(\theta,\phi)\sin\theta\right) = e^{i\phi}N_{\ell,m}P_\ell^m(\cos\theta)e^{im\phi}\cos\theta + \sin\theta\sqrt{(\ell-m)(\ell+m+1)}N_{\ell,m+1}P_\ell^{m+1}(\cos\theta)e^{i(m+1)\phi} \tag{74}$$

Substitution of Eq. (74) into Eq. (60) gives:

$$\begin{aligned} a_M(\ell,m) &= \frac{-k^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}\frac{e}{r_n^2}N \\ &\int j_\ell(kr)Y_\ell^{m*}(\theta,\phi)\delta(r-r_n)\left\{\begin{array}{l} e^{i\phi}N_{\ell,m}P_\ell^m(\cos\theta)e^{im\phi}\cos\theta \\ \\ +\sin\theta\sqrt{(\ell-m)(\ell+m+1)}N_{\ell,m+1}P_\ell^{m+1}(\cos\theta)e^{i(m+1)\phi} \end{array}\right\}d^3x \end{aligned} \tag{75}$$

Substitution of $Y_\ell^{-m}(\theta,\phi) = (-1)^m Y_\ell^{m*}(\theta,\phi)$ and Eq. (73) into Eq. (75) and integration with respect to dr gives:

$$a_M(\ell,m)=\frac{-ek^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}Nj_\ell(kr_n)$$
$$\int_0^{2\pi}\int_0^{\pi}(-1)^m N_{\ell,-m}P_\ell^{-m}(\cos\theta)e^{-im\phi}\left\{\begin{array}{l} e^{i\phi}N_{\ell,m}P_\ell^m(\cos\theta)e^{im\phi}\cos\theta \\ \\ +\sin\theta\sqrt{(\ell-m)(\ell+m+1)}N_{\ell,m+1}P_\ell^{m+1}(\cos\theta)e^{i(m+1)\phi}\end{array}\right\}\sin\theta d\theta d\phi \tag{76}$$

The integral in Eq. (76) separated in terms of $d\theta$ and $d\phi$ is:

$$a_M(\ell,m)=\frac{-ek^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}Nj_\ell(kr_n)$$
$$\int_0^{\pi}(-1)^m N_{\ell,-m}P_\ell^{-m}(\cos\theta)\left\{\begin{array}{l} N_{\ell,m}P_\ell^m(\cos\theta)\cos\theta \\ \\ +\sin\theta\sqrt{(\ell-m)(\ell+m+1)}N_{\ell,m+1}P_\ell^{m+1}(\cos\theta)\end{array}\right\}\sin\theta d\theta\int_0^{2\pi}e^{i\phi}d\phi \tag{77}$$

Consider that the $d\theta$ integral is finite and designated by Θ, then Eq. (77) is given as:

$$a_M(\ell,m)=\frac{-ek^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}Nj_\ell(kr_n)\Theta\int_0^{2\pi}e^{i\phi}d\phi \tag{78}$$

From Eq. (54), the far fields are given by:

$$\begin{aligned}\mathbf{B}&=-\frac{i}{k}a_M(\ell,m)\nabla\times g_\ell(kr)\mathbf{X}_{\ell,m}\\ \mathbf{E}&=a_M(\ell,m)g_\ell(kr)\mathbf{X}_{\ell,m}\end{aligned} \tag{79}$$

where $a_M(\ell,m)$ is given by Eq. (78).

The power density $P(t)$ given by the Poynting power vector is:

$$P(t)=\mathbf{E}\times\mathbf{H} \tag{80}$$

For a pure multipole of order (ℓ,m), the time-averaged power radiated per solid angle $\frac{dP(\ell,m)}{d\Omega}$ given by Eqs. (16.74) and (16.75) of Jackson [15] is:

$$\frac{dP(\ell,m)}{d\Omega}=\frac{c}{8\pi k^2}\left|a_M(\ell,m)\right|^2\left|\mathbf{X}_{\ell,m}\right|^2 \tag{81}$$

where $a_M(\ell,m)$ is given by Eq. (78).

The modulation function $Y_{\ell,m}(\theta,\phi)$ is a traveling charge-density wave that moves time harmonically on the surface of the orbitsphere, spins about the z-axis with frequency ω_n, and modulates at $m\omega_n$ corresponding to the term $m\omega_n t$ in Eq. (49). The independent variable ϕ is also a term of the argument of the spherical harmonic function as shown in Eq. (49). Consider the entire potentially radiating surface and the single quantized potentially emitted photon that carries all of the conserved angular momentum of $\hbar$ and energy given by Planck's equation. The time dependence of the power is eliminated in Eq. (81), but the boundary condition of the azimuthal spatial integral for $a_M(\ell,m)$ over its ϕ dependence can also be evaluated in Eqs. (78) and (81) according to the source current's space and time dependence using a substitution of variable for ϕ. From the azimuthal dependency of the source current corresponding to one period, Eq. (78) that can be written as:

$$a_M(\ell,m)=\frac{-ek^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}Nj_\ell(kr_n)\Theta\int_0^{vT_n}\cos(ks)ds \tag{82}$$

where s is the distance along a current path with the corresponding limit of integration being the angular displacement of the rotating modulation function during one period T_n at the linear velocity in the $\hat{\phi}$ direction of v, and k is the wavenumber corresponding to the angular frequency. Thus,

$$a_M(\ell,m)=\frac{-ek^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}Nj_\ell(kr_n)\Theta\sin(kvT_n) \tag{83}$$

$$a_M(\ell,m)=\frac{-ek^2}{c\sqrt{\ell(\ell+1)}}\frac{\omega_n}{2\pi}Nj_\ell(kr_n)\Theta\sin(ks) \tag{84}$$

In the case that k is the light-like k^0, then $k=\omega_n/c$, and the $\sin(ks)$ term in Eq. (84) vanishes for,

$$R=cT_n \tag{85}$$

$$RT_n^{-1}=c \tag{86}$$

$$Rf=c \tag{87}$$

Here ω_n refers to Eq. (48) regarding the angular frequency given by Eq. (1.36) corresponding to the frequency of a potentially emitted photon as given in Chp. 2. Thus,

$$s=vT_n=R=r_n=\lambda_n \tag{88}$$

as given by Eq. (1.279) which is identical to the Haus condition for nonradiation given by Eq. (43), and the photon emission condition given by Eq. (88) is equivalent to that of Eq. (67). Then, the multipole coefficient $a_M(\ell,m)$ is zero as it also has to be according to Eq. (78). For the condition given by Eq. (88), the time-averaged power radiated per solid angle $\frac{dP(\ell,m)}{d\Omega}$ given by Eqs (81) and (84) is zero. *There is no radiation*.

REFERENCES

1. H. A. Haus, "On the radiation from point charges," American Journal of Physics, 54, (1986), pp. 1126-1129.
2. R. N. Bracewell, *The Fourier Transform and Its Applications*, McGraw-Hill Book Company, New York, (1978), pp. 252-253.
3. W. McC. Siebert, *Circuits, Signals, and Systems*, The MIT Press, Cambridge, Massachusetts, (1986), p. 415.
4. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, (1962), p.22.
5. M. Abramowitz, I. Stegun (3rd Printing 1965), p. 366, Eq. 9.1.10, and p. 255, Eq. 6.1.6.
6. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, (1962), p.30.
7. H. Bateman, *Tables of Integral Transforms*, Vol. III, McGraw-Hill, New York, (1954), p. 33.
8. Personal communication from M. Nansteel February 2011, confirmed by Mathematica™ by J. Lotoski.
9. G. O. Reynolds, J. B. DeVelis, G. B. Parrent, B. J. Thompson, *The New Physical Optics Notebook,* SPIE Optical Engineering Press, (1990).
10. T. A. Abbott, D. J. Griffiths, Am. J. Phys., Vol. 53, No. 12, (1985), pp. 1203-1211.
11. P. Pearle, Foundations of Physics, "Absence of radiationless motions of relativistically rigid classical electron," Vol. 7, Nos. 11/12, (1977), pp. 931-945.
12. G. Goedecke, Phys. Rev 135B, (1964), p. 281.
13. J. Daboul and J. H. D. Jensen, Z. Physik, Vol. 265, (1973), pp. 455-478.
14. L. C. Shi, J. A. Kong, *Applied Electromagnetism*, Brooks/Cole Engineering Division, Monterey, CA, (1983), pp. 170-209.
15. J. D. Jackson, *Classical Electrodynamics*, Second Edition, John Wiley & Sons, New York, (1975), pp. 739-779.
16. G. R. Fowles, *Analytical Mechanics*, Third Edition, Holt, Rinehart, and Winston, New York, (1977), p. 35.